Math 602 HW3

1 Exercises

1. Show that xy^2-xz+y, $xy-z^2$, $x-yz^4$ is NOT a Groebner basis (if we are using the lex order) on $\mathbb{Z}^3_{\geq 0}$. You don't need to bother with Buchberger's criterion.

2. Divide $x^7y^2+x^3y^2-y+1$ by $\langle xy^2-x, x-y^3\rangle$. Use the lex order. Do the division with the generators in one order, then the other.

3. $\{y-x^2, y-z^3\}$ is NOT a Groebner basis

4. Show that in a ring R that the set of nilpotent elements, denoted $\text{nil}(A) = \sqrt{0}$ is in fact an ideal.

5. In the following two sub-exercises you'll prove the Prime avoidance theorems.

   (a) Prime Avoidance 1. Suppose $I \subseteq \bigcup_{i-1}^{n} \mathfrak{p}_i$ where $\mathfrak{p}_i$ are prime. Then $I \subset \mathfrak{p}_j$ for some j

   (b) Prime Avoidance 2. Suppose $\mathfrak{p}$ is a prime and

   $$\mathfrak{p} \supset \bigcap_{i=1}^{n} I_i$$

   then $\mathfrak{p} \supset I_j$ for some j.

6. If M, N are both flat R modules, then so is $M \otimes_R N$

7. R a ring, I an ideal, M an R module. Show that $M/IM \cong (R/I) \otimes_R M$. One way to do this is to tensor the exact sequence of R-modules $0 \to I \to R \to R/I \to 0$ with M.

8. Prove $\mathbb{Z}/(a) \otimes_{\mathbb{Z}} \mathbb{Z}/(b) = 0$ if a, b are coprime.

9. Compute $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$.

10. $0 \to M' \to M \to M'' \to 0$ an exact sequence with M, M'' finitely generated. Then M is finitely generated.

11. R a domain and K its fraction field. Let I be an ideal of R considered as an R-module. Then $I \otimes_R K = K$.

12. Do as many of Atiyah-Macdonald Ch 1. Problems 15-21 as you can (NOT TO BE TURNED IN)

2 Computation

Here are some low-stakes computational exercises juste to get you acquainted with a computer algebra system.

1. In your favorite computer algebra system, define the ring $\mathbb{Q}[x, y, z]$ with lexiographic order. Also define it woth reverse lexicographic order.

2. Seems dumb, but compute the leading terms of $4z^3 + xy^2 + 3z$ in $\mathbb{Q}[x, y, z]$ using lex and revlex.

3. Compute a Groebner basis for $\langle xy + y^2 - 1, xy \rangle$ in $\mathbb{Q}[x, y, z]$ using graded reverse lexicographic order.

3 Reading

Find a proof of Buchberger's algorithm in either Cox,Little,O'Shea or Kemper. Read it.

Read Kleiman-Altman Chapters 6 and 7. Exercises from this reading will come next week.