Math 602 HW 2

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1. Let R and S be domains and let $u \in S \setminus \{0\}$ be integral over R. Then some multiple of u (i.e. an su with $s \in S$) lives in R.

2. Let $\mathfrak{p}$ be an ideal in a ring R. Then $\mathfrak{p}$ is prime if and only if $R - \mathfrak{p}$ is a multiplicative system. Consider the statement: If R is a ring and $\mathcal{S}$ is a multiplicative system, then $R - \mathcal{S}$ is prime. Show that this isn't true by finding a counterexample.

3. Let $R \hookrightarrow S$ be an integral extension. In the proof of the "Lying over theorem", I used the following fact: Let $\mathfrak{p}$ be a prime in R and $\mathfrak{q}$ a prime in S such that $\mathfrak{q} \cap R = \mathfrak{p}$. Then $R/\mathfrak{p} \hookrightarrow S/\mathfrak{q}$ is an integral extension.

   In fact, more is true. Suppose $R \hookrightarrow S$ is an integral extension, then if $J \subset S$ is an ideal $R/(R \cap J) \to S/J$ is an integral extension.

4. Let I be an ideal of a ring R, show that there is a bijection between primes in R/I and primes in R containing I.

5. Let $R \hookrightarrow S$ be any inclusion of rings. Suppose that $\mathfrak{q}$ is prime in S. Then $R \cap \mathfrak{q}$ is prime in R

6. Let R and S be *integral domains* and suppose that $R \subset S$ is an integral extension. Show that R is a field if and only if S is a field.

7. Atiyah-Macdonald Ex. V.2. Let $R \hookrightarrow S$ be an integral extension. Suppose $f : R \to K$ is a map into an *algebraically closed* field. Then f can be extended to a map $\overline{f} : B \to K$.

8. Let $R \hookrightarrow S$ be an inclusion of rings. Show that $\overline{R}$, the set of elements in S integral over R is a ring.