# HW #1

January 10, 2020

Now, and forevermore, rings will be commutative, and with unit.

Some of the following exercises are not so interesting. But the will be useful in juggling operations on ideals, which will be essential.

## 1 Exercises

1. Show that
   - $V(I \cap J) = V(IJ) = V(I) \cup V(J)$.
   - $V(I + J) = V(I) \cap V(J)$.
   - If $I \subset J$ then $V(I) \supset V(J)$.
   - If $X \subset Y$ then $I(X) \supset I(Y)$.
   - If $Z$ is a variety, show that $Z = V(\mathcal{I}(Z))$.
2. We need to review the behavior of ideals under maps. Let $f : A \to B$ be a ring homomorphism. Let $I, I_1, I_2$ be ideals of $A$, and $J, J_1, J_2$ ideals $B$. Define the **extension** of an ideal $I$ in $A$ to be $I^e = f(I)B$. This is sometimes also denoted $IB$. Simlarly, defined the **contraction** of an ideal $J$ in $B$ to be $J^c = f^{-1}(J)$.
   - Show $(I_1 + I_2)^e = I_1^e + I_2^e$
   - Show $(I_1 \cap I_2)^e \subset I_1^e \cap I_2^e$
   - Show $(I_1 I_2)^e = I_1^e I_2^e$
   - Show $(J_1 + J_2)^c \supset J_1^c + J_2^c$
   - Show $(J_1 \cap J_2)^c = J_1^c \cap J_2^c$
   - Show $(J_1 J_2)^c \supset J_1^c J_2^c$
3. Though we don't have any motivation for this construction yet, we define the **colon ideal**. Let $I, J \subset R$ be ideals. Define

   $$(I : J) = \{r \in R : |rJ \subset I\}$$

   Verify that this is indeed an ideal. Verify these identities for ideals $I, J, K$ in $R$

- $I \subset (I : J)$
- $(I : J)J \subset I$
- $((I : J) : K) = (I : JK) = ((I : K) : J)$
- $(\bigcap_i I_i : J) = \bigcap_i (I_i : J)$
- $(I : \sum_i J_i) = \bigcap_i (I : J_i)$.

Then verify the following for ideals (with the same notation as the previous problem)

- $(I_1 : I_2)^e \subset (I_1^e : I_2^e)$
- $(J_1 : J_2)^c \subset (J_1^c : J_2^c)$

4. Some review on prime and maximal ideals. Let $\mathfrak{p}$ be a prime ideal in a ring $R$. Show that $\mathfrak{p}[x]$ is a prime ideal in $R[x]$. If $\mathfrak{m}$ is a maximal ideal in $R$, is $\mathfrak{m}[x]$ necessarily a maximal ieal in $R[x]$ (Hint: quotient characterization).

5. **The Chinese Remainder Theorem**. In steps.

   (a) $I, J$ ideals in $R$ are **comaximal** if $I + J = R = (1)$. Show
      i. $IJ = I \cap J$
      ii. $R/IJ \cong R/I \times R/J$

   (b) Let $I$ be comaximal to both $J, J'$. Then $I$ comaximal to $J, J'$.

   (c) Show $I, J$ are comaximal if and only if $I^m$ and $J^n$ are (for any $m, n \geq 1$)

   (d) $I_1, \ldots, I_n$ pairwise comaximal. Show
      - $I_1$ and $I_1 \cdots I_n$ are comaximal
      - $I_1 \cap \cdots \cap I_n = I_1 \cdots I_n$
      - $R/(I_1 \cdots I_n) \xrightarrow{\cong} \prod_i R/I_i$

   This is also sometimes stated as

$$R/\bigcap I_i \xrightarrow{\cong} \prod_i R/I_i$$

   for $I_i$ comaximal. Look the Chinese Remainder Theorem up on wikipedia to see why this somewhat dry theorem I just stated is call the Chinese Remainder Theorem.

6. An element $x$ of a ring $R$ is **nilpotent** if $x^n = 0$. Show that $1 + x$ is a unit in $R$.

# 2 Reading

From Atiyah-Macdonald, read Ch. 2. We will go through most of this later, but it would be good to look at it now. Also read sections 4 and 5 in "A Term of Commutative Algebra."