Math 602: Homological Algebra Edition

1 Introduction

This document is meant to be an exercise-based introduction to homological algebra. There is really no other way to do it — anything else is to dull. One just has to do it onesself.

2 Basics

We define the following concepts, which you will work with in the rest of this worksheet / long homework assignment.

Definition 2.1.

A sequence of R-modules

$$0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$$

is **(short) exact** if f is injective, $\ker g = \operatorname{im} f$ and g is surjective.

Definition 2.2. A sequence of R-modules

$$\cdots \to M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} \cdots$$

is **exact** if $\operatorname{im}(d_i) = \ker(d_{i-1})$

Definition 2.3. A sequence of R-modules

$$\cdots \to M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} \cdots$$

such that $d_{i-1} \cdot d_i = 0$ (often abbreviated $d^2 = 0$) is a **chain complex**.

Remark 2.4. In what follows, we will often abbreviate every differential in a chain complex as "d" with the understanding that if $d : C_i \to C_{i-1}$ obviously means d_i. If there are multiple chain complexes in the mix, sometimes we will use d_C.

What is the motivation for these? Exact sequence just seem to arise in nature. A more high-brow explanation is that the theory of chain complexes is really the *homotopy theory* of modules. This statement is not just wish-washy intuition, it can actually be made precise via model categories. Chain complexes also arise natural in homology/cohomology.

Definition 2.5. A **morphism of chain complexes** $f : C_\bullet \to D_\bullet$ is given by a collection of maps $f_i : C_i \to D_i$ making the following diagram commute:

$$\begin{array}{ccccc} C_{i+1} & \longrightarrow & C_i & \longrightarrow & C_{i-1} \\ \downarrow & & \downarrow & & \downarrow \\ D_{i+1} & \longrightarrow & D_i & \longrightarrow & D_{i-1} \end{array}$$

3 Diagram Chases: Sudoku, but with arrows

Each of the lemmas below are "standard" and often get invoked in mathematical literature without any warning or notice. It is worthwhile to internalize them and to really see why they work.

EXERCISE 3.1 (The Snake Lemma). Given a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\ 0 & \longrightarrow & N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 & \longrightarrow & 0 \end{array}$$

show that there is an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker(f_1) & \longrightarrow & \ker(f_2) & \longrightarrow & \ker(f_3) & & \\ & & & & & \searrow & & & \\ & & \operatorname{coker}(f_1) & \longrightarrow & \operatorname{coker}(f_2) & \longrightarrow & \operatorname{coker}(f_3) & \longrightarrow & 0 \end{array}$$

EXERCISE 3.2 (The Five Lemma). Consider the diagram

$$\begin{array}{ccccccccc} M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & M_4 & \longrightarrow & M_5 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\ N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 & \longrightarrow & N_4 & \longrightarrow & N_5 \end{array}$$

with exact rows.

Show that

1.

EXERCISE 3.3 (The Nine Lemma). Consider the following commutative diagram where all columns are exact and the middle row is exact.

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & M & \longrightarrow & N & \longrightarrow & P & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M' & \longrightarrow & N' & \longrightarrow & P' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & M'' & \longrightarrow & N'' & \longrightarrow & P'' & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Prove that the first row is exact if and only if the third row is exact.

4 Some Exactness Properties

EXERCISE 4.1. That is, show that if

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$$

is exact, then

$$M \otimes_R M' \longrightarrow N \otimes_R M' \longrightarrow P \otimes_R M' \longrightarrow 0$$

is exact. Notice that it is NOT exact on the left.

Hom has less clear exactness properties because of variance properties. Nevertheless, show the following.

EXERCISE 4.2. Let M be an R-module and

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$$

Show that

$$0 \longrightarrow \hom_R(M, N') \longrightarrow \hom_R(M, N) \longrightarrow \hom_R(M, N'')$$

is exact

EXERCISE 4.3. Let N be an R-module and

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is exact. Show that

$$0 \longrightarrow \hom_R(M'', N) \longrightarrow \hom_R(M, N) \longrightarrow \hom_R(M', N)$$

is exact.

The cases where exactness for tensor and Hom occur are important — thesse represent flatness and Injective and Projective modules

5 Basic Flatness / Faithful Flatness

Recall that an R-module is flat if the functor $M \otimes_R -$ is an exact functor. This means that $M \otimes_R -$ preserves exact sequences. There is also another concept which will be useful.

Definition 5.1. An R-module M is **faithfully flat** is M also *reflects* exact sequences. That is $0 \to X \to Y \to Z \to 0$ is exact IF AND ONLY IF

$$0 \longrightarrow M \otimes_R X \longrightarrow M \otimes_R Y \longrightarrow M \otimes_R Z \longrightarrow 0$$

is exact.

Definition 5.2. A morphism $f : R \to S$ of rings is **flat** if it gives S the structure of a flat R-module. One also says that B **is a flat** A**-algebra**.

EXERCISE 5.3 (Flatness is Local). Let M be an R-module. Show that the following are equivalence

1. M is a flat R-module
2. $M_{\mathfrak{p}}$ is a flat $R_{\mathfrak{p}}$ module for each $\mathfrak{p}$ prime in R
3. $M_{\mathfrak{m}}$ is a flat $R_{\mathfrak{m}}$ module for each $\mathfrak{m}$ maximal in R.

The hard part is 3) implies 1). To help, note that flatness is equivalent to preserving *injectivity* (because preserving surjectivity comes for free).

EXERCISE 5.4 (Free $\Rightarrow$ Flat). Let F be a free R-module. Show that F is flat.

The following exercise should be done after doing Sect. 6.

EXERCISE 5.5 (Projective $\Rightarrow$ Flat). Show that projective modules are flat.

EXERCISE 5.6 (Basisc Transitivity for flatness / faithful flatness).

EXERCISE 5.7.

6 Injective, Projective

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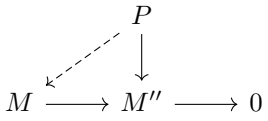
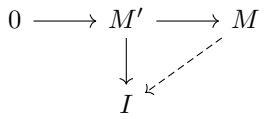
Injective and projective modules have pleasant catgorical properties that allow for easy diagrammatic manipulations. Projective modules are the algebraic equivalent of vector bundles and are intensely useful in homological computations. Injective modules are also important, but their theory is more subtle.

We recall the following definitions.

Definition 6.1. An R-module P is **projective** if the indicated lift always exist



Definition 6.2. An R-module I is **injective** if the indicated lift always exists



EXERCISE 6.3 (Equivalent Characterization of Projective). Show that the following are equivalent

1. P is projective
2. $\hom(P, -)$ is an exact functor
3. Every exact sequence $0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$ splits.
4. There is a module Q such that $P \oplus Q$ is a free module, that is $P \oplus Q \cong R^{\mathcal{I}}$ for some indexing

EXERCISE 6.4 (Examples of Projectives). Show that every free R-module is projective.

EXERCISE 6.5 (Characterization of Injectives). Show that the following are equivalent

1. I is injective
2. $\hom(-, I)$ is exact
3. Every exact sequence $0 \to I \to M \to M'' \to 0$ splits

7 Homology and Chain Homotopies

Let $C_\bullet$ be a chain complex

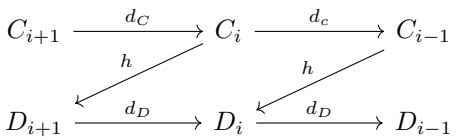
Definition 7.1. The **homology** of a chain complex is $H_i(C_\bullet) = \ker d_{i-1} / \operatorname{im} d_i$

EXERCISE 7.2. Let $f : C_\bullet \to D_\bullet$ be a map of chain complexes. Show that there is a well-defined induced map $f_* : H_*(C_\bullet) \to H_*(D_\bullet)$.

Definition 7.3. Let $f : C_\bullet \to D_\bullet$ be two maps of chain complexes. **chain homotopy** is a collection of maps $h_i : C_i \to D_{i+1}$ such that

$$f_i - g_i = d_D h + h d_C$$

(I have omitted the indices on the right, but there is only one choice that makes sense).



Remark 7.4. We say that f is **nullhomotopic** if it is homotopic to 0.

EXERCISE 7.5. Show that if there is a chain homotopy between f and g they induce the same map on homology.

The notion of chain homotopy allows us to define a notion of equivalence between chain complexes.

Definition 7.6. Two complexes $C_\bullet, D_\bullet$ are *chain homotopy equivalent* if there are $f : C_\bullet \to D_\bullet$ and $g : D_\bullet \to C_\bullet$ such that $f \circ g$ is homotopic to Id_D and $g \circ f$ is homotopic to Id_C.

There is another notion of equivalence which is weaker

Definition 7.7. $C_\bullet$ and $D_\bullet$ are **quasiequivalent** if there is a map $C_\bullet \to D_\bullet$ that induces an isomorphism $H_*(C_\bullet) \to H_*(D_\bullet)$

Remark 7.8. Being quasiequivalent and being chain homotopic are *not the same thing.* Quasiequivalence is weaker.

The following exercise is used extremely frequently in homological algebra and algebraic topology, frequently without comment.

EXERCISE 7.9. Let $0 \to C_\bullet \xrightarrow{i} D_\bullet \xrightarrow{\pi} E_\bullet \to 0$ be an exact sequence of complexes, then there is an induced **long exact sequence in homology**:

$$\begin{array}{ccccc} H_n(C_\bullet) & \xrightarrow{i_*} & H_n(D_\bullet) & \xrightarrow{\pi_*} & H_n(E_\bullet) \\ & & & \overset{\delta}{\swarrow} & \\ H_{n-1}(C_\bullet) & \xrightarrow[i_*]{} & H_{n-1}(D_\bullet) \cdots & & \end{array}$$

The hard part is defining the map δ.

8 Resolutions

Definition 8.1. A **resolution** of an R-module M is an exact sequence

Definition 8.2. A **projective resolution** of an R-module M is an exact sequence

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

such that each P_i is projective.

Remark 8.3. One often uses the shorthand $P_\bullet \to M$.

Definition 8.4. A **injective resolution** is similarly a diagram

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow \cdots$$

In what follows, the following will be useful

EXERCISE 8.5.

The intuition about (projective) resolutions is that they are "fattened" versions of modules. They key technical point is that maps between modules can be lifted to maps between resolutions.

EXERCISE 8.6 (Maps Between Projective Resolutions). Let $P_\bullet \to M$ and $P'_\bullet \to M'$. Suppose we have a map $f\colon M \to M'$. Show that there is an induced map $P_\bullet \to P'_\bullet$. That is, we can produce lifts in the following diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f} & & \\ \cdots & \longrightarrow & P'_1 & \longrightarrow & P'_0 & \longrightarrow & M' & \longrightarrow & 0 \end{array}$$

EXERCISE 8.7. Formulate and prove the corresponding statement for injective resolutions.

Going through the exercise above, you'll notice that you made many, many choices, all of them dictated by exactness. The following exercise shows you that, in some sense, the choices to do not matter.

EXERCISE 8.8 (Homotopies Between Projective Resolutions). Let $P_\bullet \to M$ and $P'_\bullet \to M'$ and consider $f\colon M \to M'$ as above. Show that any two lifts of f are homotopic. [Hint: Reduce to showing that any lift of the zero homomorphism is nullhomotopic]

EXERCISE 8.9 (The Horseshoe Lemma: resolutions of exact sequences). Consider the following diagram:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & P'_1 & \longrightarrow & P'_0 & \longrightarrow & M' & \longrightarrow & 0 \\ & & & & & & \downarrow & & \\ & & & & & & M & & \\ & & & & & & \downarrow & & \\ \cdots & \longrightarrow & P''_1 & \longrightarrow & P''_0 & \longrightarrow & M'' & \longrightarrow & 0 \end{array}$$

where $M' \to M \to M''$ is exact. Show that the middle row can be completed to a resolution. In fact, the ith stage is $P'_i \oplus P''_i$ — the hard part is producing the differentials.

9 Injective Modules

Above, we saw that any module as a projective resolution. This comes fairly easily because free modules are projective. For injective modules this is quite a bit trickier.

EXERCISE 9.1. Show that the product of injective modules in injective.

EXERCISE 9.2 (Baer's Criterion). Baer's criterion says that to check the injectivity of a module, one only has to know how the module behaves with respect to *ideals* of a ring R. That is, the following states are equivalent for an R-module M

- M is injective
- For any ideal $I \subset R$ and module map $I \to M$ there is an extension as in the following diagram

$$\begin{array}{ccc} I & \longrightarrow & M \\ \downarrow & \nearrow & \\ R & & \end{array}$$

We show the equivalence in a series of exercises

1. Show that 1) $\Rightarrow$ 2) above.
2. We show 2) $\Rightarrow$ 1) in steps. We want to show that M is injective, i.e. we want to produce a lift

$$\begin{array}{ccccc} 0 & \longrightarrow & N & \longrightarrow & N' \\ & & \downarrow & \swarrow & \\ & & M & & \end{array}$$

It was very easy to show that every module has a projective resolution. Free modules are, in particular, projective, and every module has a free resolution. Showing that every module has an injective resolution is significantly trickier.

We would like very module M to have an injective map into an injective module, I. That is, we seek $0 \to M \to I$. There will be some difficulty in what follows in type-checking: we will speak of injective maps and injective modules, which are quite distinct concepts.

EXERCISE 9.3 (Enough injectives).

10 Tensor Products of Chain Complexes

Definition 10.1. A **double complex** is a chain complexes of chain complexes. That is, it is a commutative diagram

$$\begin{array}{ccccccc}
\vdots & \vdots & \vdots & \vdots & \ddots \\
\downarrow{\scriptstyle d^v} & \downarrow{\scriptstyle d^v} & \downarrow{\scriptstyle d^v} & \downarrow{\scriptstyle d^v} \\
C_{02} & \xleftarrow[d^h]{} & C_{12} & \xleftarrow[d^h]{} & C_{22} & \xleftarrow[d^h]{} & C_{32} & \xleftarrow[d^h]{} & \cdots \\
\downarrow{\scriptstyle d^v} & & \downarrow{\scriptstyle d^v} & & \downarrow{\scriptstyle d^v} & & \downarrow{\scriptstyle d^v} \\
C_{01} & \xleftarrow[d^h]{} & C_{11} & \xleftarrow[d^h]{} & C_{21} & \xleftarrow[d^h]{} & C_{31} & \xleftarrow[d^h]{} & \cdots \\
\downarrow{\scriptstyle d^v} & & \downarrow{\scriptstyle d^v} & & \downarrow{\scriptstyle d^v} & & \downarrow{\scriptstyle d^v} \\
C_{00} & \xleftarrow[d^h]{} & C_{10} & \xleftarrow[d^h]{} & C_{20} & \xleftarrow[d^h]{} & C_{30} & \xleftarrow[d^h]{} & \cdots
\end{array}$$

Remark 10.2. Double complexes are allowed to have negatively indexed entries, but I'm using the above definition to make it easier on us.

A double complex can be "flattened" into a single complex.

Definition 10.3. The **total complex** of a double complex is defined to be

$$\mathrm{Tot}(C_{\bullet\bullet})_n = \bigoplus_{i+j=n} C_{ij}$$

with differential

$$d^{\mathrm{tot}} \colon \mathrm{Tot}(C_{\bullet\bullet})_n \to \mathrm{Tot}(C_{\bullet,\bullet})_{n-1} \qquad d^{\mathrm{tot}}(c_{ij}) = ((-1)^j d^h, d^v) \in C_{i-1,j} \oplus C_{i,j-1}$$

Remark 10.4. Note that it is enough to define the differential on C_{ij} for each i, j by the universal property of direct sums.

EXERCISE 10.5 (Double Complexes). Check that $\mathrm{Tot}(C_{\bullet,\bullet})$ is in fact a chain complex.

Remark 10.6. This construction seems somewhat *ad hoc*. However, we will see below in Sect **??** that corresponds to a very useful categorical construction.

Let $C_\bullet$ and $D_\bullet$ be chain complexes. One defines $C_\bullet \otimes_R D_\bullet$ in the only sensible way possible. You form a grid

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \ddots \\
 & & & & & & \\
C_2 \otimes D_0 & \longleftarrow & C_1 \otimes D_2 & \longleftarrow & C_2 \otimes D_2 & & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
C_1 \otimes D_0 & \longleftarrow & C_1 \otimes D_1 & \longleftarrow & C_2 \otimes D_1 & & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
C_0 \otimes D_0 & \longleftarrow & C_1 \otimes D_0 & \longleftarrow & C_2 \otimes D_1 & & \cdots
\end{array}$$

where the unlabelled arrows are either $d_C \otimes 1$ or $1 \otimes d_D$.

Definition 10.7. Let $C_\bullet, D_\bullet$ be chain complexes. One defines the tensor product of two chain complexes to be the chain complex

$$(C \otimes D)_\bullet = \mathrm{Tot}(C_\bullet \otimes D_\bullet)$$

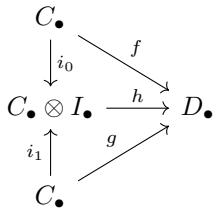
As an exercise, we use this defintion to reformulate chain homotopy. First, we need a definition.

Definition 10.8. Let $I_\bullet$ be the chain complex

$$R \longrightarrow R \oplus R$$

EXERCISE 10.9. I've called this complex $I_\bullet$ for "interval." Why does this make sense?

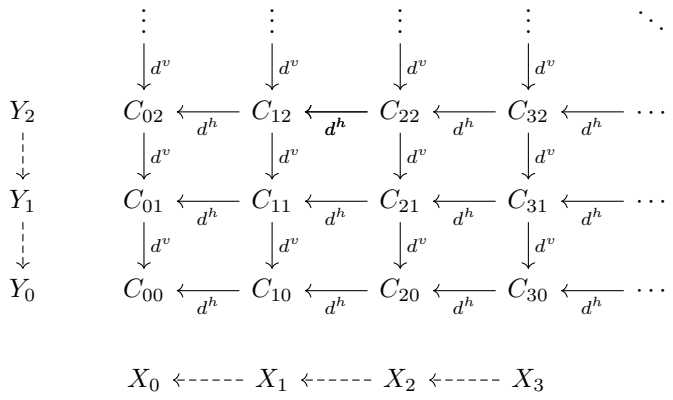
EXERCISE 10.10. Show that a homotopy between maps $f, g : C_\bullet \to D_\bullet$ is the same as specifying a diagram



[Part of this exercise is to specify what i_0 and i_1 should be]

The following exercise will be useful in the next section, and is useful in general. It is a prelude to what's called the "Double Complex Spectral Sequence." It is also a great deal of fun.

EXERCISE 10.11. Consider a double complex



where

1. The rows and columns of C_{ij} are exact
2. We define

$$X_i = \mathrm{coker}(C_{i,1} \xrightarrow{d^v} C_{i,0})$$
$$Y_i = \mathrm{coker}(C_{1,i} \xrightarrow{d^h} C_{0,i})$$

Note that there are then canonical maps $\pi\colon C_{n,0} \to X_n$ and $\pi\colon C_{0,n} \to Y_n$ given by projection.

Now, we do the following steps:

1. Show $X_\bullet$ and $Y_\bullet$ have the structure of chain complexes, with differentials induced by appropriate maps in C_{ij}

2. Define a map $\Phi : \mathrm{Tot}(C_{\bullet,\bullet}) \to X_\bullet$ by

$$\Phi_n : \mathrm{Tot}(C_{\bullet,\bullet})_n \to X_n \qquad \Phi_n\left(\sum_{i=0}^{n} c_{i,n-i}\right) = \pi(c_{n,0})$$

   Check that Φ is indeed a map of chain complexes.

3. Prove that Φ is a quasievuivalence (that it is, induces an isomorphism on homology) [Hint: Prove surjectivity and injectivity separately — their proofs are very similar. Your proof should use a lot of zig-zagging around]

4. Show that we obtain an abstract isomorphism $H_*(Y_\bullet) \cong H_*(X_\bullet)$

11 Tor

This is possibly your first example of a "derived functor." Below, I'll try to give you a flavor of what a derived functor really is. For now, we use the following motivation .We know that tensor product is right exact. That is, if $0 \to M' \to M \to M'' \to 0$ is exact, then only

$$M' \otimes_R N \longrightarrow M \otimes_R N \longrightarrow M'' \otimes_R N \longrightarrow 0$$

Definition 11.1. Let M, N be R-modules. Let $P_\bullet \to M$ be a resolution of M. Apply the functor $- \otimes_R N$ to $P_\bullet$:

$$P_2 \otimes_R N \longrightarrow P_1 \otimes_R N \longrightarrow P_0 \otimes_R N$$

The homology of this chain complex is called $\mathrm{Tor}_R^i(M, N)$

EXERCISE 11.2. Check that $\mathrm{Tor}_R^0(M, N) \cong M \otimes_R N$

EXERCISE 11.3. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of R-modules and let N also e an R-module. Show that there is a long exact sequence

$$\begin{array}{ccccc} \cdots & & \mathrm{Tor}_R^2(M,N) & & \mathrm{Tor}_R^2(M'',N) \\ & \searrow & & \nearrow & \\ \mathrm{Tor}^1(M',N) & \longrightarrow & \mathrm{Tor}^1(M,N) & \longrightarrow & \mathrm{Tor}^1(M'',N) \\ & \searrow & & \nearrow & \\ M' \otimes_R N & \longrightarrow & M \otimes_R N & \longrightarrow & M'' \otimes_R N \end{array}$$

[Hint: horseshoe lemma, long exact sequences in homology]

EXERCISE 11.4. Show that $\mathrm{Tor}_R^n(M, N) \cong \mathrm{Tor}_R^n(N, M)$.

EXERCISE 11.5. Using Ex. 11.4 redo Ex. 11.3 by resolving the other variable.

12 Some More Exercises on Flatness

We now have enough tools to prove everything we could ever want about flatness (well, sort of...we can prove a lot of things).

EXERCISE 12.1 (Tor vanishing). If M, N is flat then $\mathrm{Tor}_i^R(M, N) = 0$ for $i > 0$

EXERCISE 12.2 (Flat Commutativity w/ Tor). If B is a flat A-algebra, then

$$B \otimes_A \operatorname{Tor}_A^i(M,N) = \operatorname{Tor}_A^i(B \otimes_A M, B \otimes_A N)$$

EXERCISE 12.3. Let $S \subset A$ be a multiplicatvely closed subset. Show that

$$S^{-1}A \otimes_A \operatorname{Tor}_A^i(M,N) \cong \operatorname{Tor}_A^i(S^{-1}M, S^{-1}N)$$

EXERCISE 12.4 (Eisenbud, p. 162, but no peeking). Let R be a ring, $x \in R$ a nonzerodivisor. Let M be an R-module. Show

$$\operatorname{Tor}_R^i(R/(x), M) = \begin{cases} M/xM & i = 0 \\ (0:x) & i = 1 \\ 0 & i > 1 \end{cases}$$

Remark 12.5. This is a baby example of what one can do with Koszul complexes.

EXERCISE 12.6.

EXERCISE 12.7 (Flat Base Change and Hom).

13 Ext

Due to injective modules having different properties than projective modules, we have separated the concepts of Tor and Ext.

Definition 13.1.

Here is an alternate definition

14 Derived Functors

In many categories in mathematics, there is a notion of "equivalence" that is weaker than the categorical notion of isomorphism. For example, in topology, one encounters the notion of homotopy equivalence. In homological algebra, one encounteres the notion of "quasiequivalence." The issue is that these weaker notions of equivalence do not necessarily preserve the constructions that you might like them to.

15 Categorical Constructions

Total complexes and total homotopy cofibers.

Definition 15.1. Quotient.

Definition 15.2. Pushout

The pushout has a serious issue. In order to illustrate it we need a little fact

EXERCISE 15.3. Let $C_\bullet$ be a chain complex. Check that $C_\bullet \otimes I_\bullet$ is nullhomotopic.

EXERCISE 15.4. Show that the pushout

First, we recall a construction from topology. Let $X \to Y$ be maps of topological spaces.

Definition 15.5. The **mapping cylinder** of f

Definition 15.6. The **mapping cone** is the following construction:

$$\Big((X \times I) \amalg Y\Big) \Big/ (x_1, 0) \sim (x_2, 0) \quad (x, 1) \sim f(x).$$

Note that one of the main properties of this is that $Y \to C(f)$ is a cofibration. The quotient $C(f) = \Sigma X$. This allows us to keep going

$$X \xrightarrow{f} Y \longrightarrow C(f) \longrightarrow \Sigma X \longrightarrow \Sigma Y$$

Definition 15.7. Define the **mapping cylinder** of $C_\bullet \to D_\bullet$ by the pushout

$$\begin{array}{ccc} C_\bullet & \longrightarrow & D_\bullet \\ \downarrow{\scriptstyle i_0} & & \downarrow \\ C_\bullet \otimes I_\bullet & \dashrightarrow & M(f) \end{array}$$

Definition 15.8. Let $C_\bullet \to D_\bullet$ be a map of chain complexes. Define the **mapping cone** $C(f)_\bullet$ as follows

$$C(f)_n = C_n \oplus D_{n-1}$$

We present the differential as a matrix

$$\begin{pmatrix} d_C & f \\ 0 & -d_D \end{pmatrix}$$

Definition 15.9. Check the following

EXERCISE 15.10. There is another way of presenting the mapping cone which is slightly easier to remember Consider the double complex $D_{\bullet,\bullet}$ (with zeros everywhere above the indicated entries).

$$\begin{array}{ccccccc} C_0 & \longleftarrow & C_1 & \longleftarrow & C_2 & \longleftarrow & C_3 & \cdots \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & \\ D_0 & \longleftarrow & D_1 & \longleftarrow & D_2 & \longleftarrow & D_3 & \cdots \end{array}$$

Show that

$$\mathrm{Tot}(D_{\bullet,\bullet}) = C(f)$$

16 Koszul Complexes

Koszul complexes become important in homological methods of dimension theory in commutative algebra.

Our first introduction to this is the following mild exercise.

EXERCISE 16.1. Let $r \in R$. Consider the following complex

$$R \xrightarrow{\cdot r} R.$$

Show that it is the cofiber of the map $r\cdot : R \to R$ where we consider each ring as a trivial chain complex supported in degree 0 [Hint: this is almost tautological, but that's fine]

Definition 16.2. Call the complex in the previous exercise $K(r)$.

EXERCISE 16.3. Let M be an R-module, and consider the complex

$$M \xrightarrow{\cdot r} M.$$

Show that this complex is $K(r) \otimes M$ (again, we consider M as a trivial complex).