Math 602 HW5

February 28, 2020

1 Exercises

1. Let $\alpha_1, \ldots, \alpha_m$ be elements in any ring. We define the **vanderMonde matrix** to be

$$A = \begin{pmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{m-1} \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{m-1} \\ & & \ddots & & \\ 1 & \alpha_m & \alpha_m^2 & \cdots & \alpha_m^m \end{pmatrix}$$

Show that

$$\Delta := \det(A) = \prod_{1 \le i < j \le m}^{m} (\alpha_j - \alpha_i)$$

and that

$$D := \Delta^2 = \prod_{i,j}^{m} (\alpha_j - \alpha_i)$$

2. (Finiteness of Integral Closure) We prove, in steps, the following theorem **Thm:** Let A be an integrally closed domain, K its fraction field. Let L/K be a finite, separable extension of K. Let B denote the integral closure of A in L. Then B is finite as an A-module

$$\begin{array}{ccc} B & \longrightarrow & L \\ \vdots & & \vdots \\ \downarrow & & \downarrow \\ A & \longrightarrow & K \end{array}$$

   (a) Show that we can reduce to the case where L is Galois [Hint: theorem of primitive element]

   (b) Let $L = K[\alpha]$ with L/K Galois. Let $G = \mathrm{Gal}(L/K)$. Show that we can choose α to be integral over A, so that $\alpha \in B$.

   (c) If $b \in B$ show that for any $\sigma \in \mathrm{Gal}(L/K)$, $\sigma(b) \in B$.

(d) Now, given $b \in B$ note that it has the form $b = \sum k_i \alpha^i$. Letting $\mathrm{Gal}(L/K) = \{\sigma_1, \ldots, \sigma_n\}$ be some ordering of the Galois group. Then note

$$\sigma_j \left(\sum k_i \alpha^i \right) = \sum k_i \sigma_j(\alpha)^i$$

so that the action of the Galois group on B can be encoded as a matrix

$$\begin{pmatrix} 1 & \sigma_1(\alpha) & \sigma_1(\alpha)^2 & \cdots & \sigma_1(\alpha)^{n-1} \\ 1 & \sigma_2(\alpha) & \sigma_2(\alpha)^{n-1} & \cdots & \sigma_2(\alpha)^{n-1} \\ \cdots & \cdots & \cdots & \ddots & \vdots \\ 1 & \sigma_n(\alpha) & \sigma_n(\alpha)^2 & \cdots & \sigma_n(\alpha)^{n-1} \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} = \begin{pmatrix} \sigma_1(b) \\ \sigma_2(b) \\ \vdots \\ \sigma_n(b) \end{pmatrix}$$

Show that if Δ denotes the determinant of the matrix on the left, then $\Delta^2 k_j \in B$ for each k_j. [Hint: adjugate matrix]

(e) Compute Δ^2 [Hint: exercise 1]

(f) Show that $\Delta^2 \in K$. Use the separability of α to show $\Delta^2 \neq 0$.

(g) Show that $\Delta^2 k_j \in A$.

(h) Show that

$$\Delta^2 B \subset A \oplus A\alpha \oplus A\alpha^2 \cdots \oplus A\alpha^{n-1}$$

so that B is finitely generated.

3. (Artin-Tate Lemma) The following lemma is used by Artin and Tate in a very clever proof of the Nullstellensatz. It's good practice for finiteness conditions. Suppose $R \subset S \subset T$ are rings. Suppose that R is Noetherian, and T is finitely generated as an A-module. Suppose further that C is finitely generated as a B-*module*. Then B is finitely generated as an A-algebra.

4. (Gauss) Every UFD is integrally closed.

5. Show that in a Noetherian ring R every ideal I contains some power of its radical $\sqrt{I}$. That is, there is some n such that $(\sqrt{I})^n \subset I$.

6. Let $\mathfrak{q}$ be primary. Show that $\sqrt{q}$ is prime. Show also that the contraction of a primary ideal is primary.

7. Let $\mathfrak{m}$ be a maximal ideal. Show that $\mathfrak{m}^n$ is primary. More generally, if $\sqrt{I}$ is maximal, then I is primary. However, it is not always true that if $\mathfrak{p}$ is prime then $\mathfrak{p}^n$ is primary.

8. Let $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ be the primes associated to the ideals (0) in a Noetherian ring. Show that $\bigcup_{i=1}^n \mathfrak{p}_i$ is the set of zero divisors.

9. If $\mathfrak{q}$ is is a $\mathfrak{p}$-primary ideal and $b \notin \mathfrak{q}$ then $(\mathfrak{q} : b)$ is $\mathfrak{p}$-primary.

10. Prove the Splitting Theorem for modules: Let R be a Noetherian ring and M a finitely generated R-module. Let $N \subset M$ be an R module. Suppose $xm \in N$ for some $m \notin N$ and further suppose $x^n \notin \mathrm{Ann}(M/N)$. Then

$$N = \{m : x^n m \in M\} \cap (x^n M + N)$$

2 Computation

For later use, we want to be able to easily compute radicals of monomial ideals. Use your favorite computer algebra system to compute some examples of radicals of monomial ideals. Guess the correct rule, and then prove your rule correct.

Rmk Don't bother turning this in, but do it. It's good for the soul, and we'll need it later.

3 Reading

You have enough information to read the treatment of primary decomposition in any textbook. Go find as many textbooks / lecture notes as you can with a treatment of primary decomposition try to make them all cohere.