Math 602 HW4

February 16, 2020

1 Exercises

1. Let A be a ring and S a multiplicative system. Show that
$$S^{-1}\sqrt{I} = \sqrt{S^{-1}I}$$

2. Let R be a commutative ring, and consider $\mathrm{Spec}(R)$. Show that $\mathrm{Spec}(R)$ is *quasicompact* (i.e. every open cover has a finite subcover). To do this, you only need to know the following: The open sets $\mathrm{Spec}(R) \setminus V(f)$ form a basis for the topology of $\mathrm{Spec}(R)$.

3. Let R be an integral domain and $I \subset A$ an ideal. Prove that I is a free A-module if and only if I can be generated by one element.

4. Let R be a ring and and M an R-module. Suppose that f_i is a set of elements in R that generate the unit ideal, i.e. $\langle f_1, \dots, f_n \rangle = R$. Show the following

   (a) If $m \in M$ becomes 0 in each M_{f_i} then $m = 0$ (M_{f_i} denotes the module localized via the multiplicative system $\{1, f_i, f_i^2, \dots\}$)

   (b) Suppose we are given elements $m_i \in M_{f_i}$ such that the image of $m_i \in M_{f_i f_j}$ and the image of m_j in $M_{f_i f_j}$ agree. Then there is an element $m \in M$ such that the image of m in each M_{f_i} is m_i.

   This is essentially a partition of unity. Note that by the first part, the answer in the second part is unique.

5. Let M be an R module and I an ideal of A. Suppose $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m}$ containing I. Prove that $M = IM$.

6. For two R-modules, define
$$(M : N) = \{r \in R : rN \subset M\} \subseteq R$$
   Note that this is nonempty since $0 \in (M : N)$. Note that this coincides with the definition of colon ideals if we consider ideals I, J as modules.

(a) Show that $(M : N)$ is an ideal

(b) Show that
$$(M : N) = \mathrm{Ann}((M + N)/M)$$

(c) Use results from class to show that if I, J are ideals then
$$S^{-1}(I : J) = (S^{-1}I : S^{-1}J)$$
if J is finitely generated.

7. R a ring and M an R-module. Let $\mathfrak{p} \in \mathrm{Supp}(M)$. Show that
$$V(\mathfrak{p}) \subset \mathrm{Supp}(M)$$

8. Let $A \subseteq B$ be rings with B integral over A. Let S be a multiplicatively closed subset of A. Then $S^{-1}B$ is integral over $S^{-1}A$. This is quite important and will be used often.